

# Lie point symmetry analysis of a second order differential equation with singularity

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## Abstract

By using Lie symmetry methods, we identify a class of second order non-linear ordinary differential equations invariant under at least one dimensional subgroup of the symmetry group of the Ermakov-Pinney equation. In this context, nonlinear superposition rule for second order Kummer-Schwarz equation is rediscovered. Invariance under one-dimensional symmetry group is also used to obtain first integrals (Ermakov-Lewis invariants). Our motivation is a type of equations with singular term that arises in many applications, in particular in the study of general NLS (nonlinear Schrödinger) equations.

## 1 Introduction

The primary motivation of this paper is to analyze the Lie point symmetries of the second order differential equation with cubic singularity

$$\ddot{x}(t) + p(t)x(t) = qx(t)^{-3} + g(t)x(t)^m, \quad (1.1)$$

where  $q, m$  are constants and  $p(t)$  and  $g(t)$  are arbitrary functions of  $t$ . As we will see in Section 6, such equation arises in a variety of applications coming from Classical and Quantum Mechanics and Geometry.

In the absence of the singular term ( $q = 0$ ), the study of Lie symmetries has been performed in previous works (see [1–3]). On the other hand, when  $g(t) \equiv 0$ , we have the familiar Ermakov-Pinney (EP) equation

$$\ddot{x}(t) + p(t)x(t) = qx(t)^{-3}. \quad (1.2)$$

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EP equation is known to be invariant under the special linear group  $SL(2, \mathbb{R})$  and admit a general solution formula (a nonlinear superposition) in terms of linearly independent solutions of the time-dependent oscillator equation

$$\ddot{x} + p(t)x = 0. \quad (1.3)$$

In a very short paper [4], Pinney presented its general solution as

$$x(t) = (Au^2 + 2Buv + Cv^2)^{1/2}, \quad (AC - B^2)W^2 = q \quad (1.4)$$

where  $u$  and  $v$  are two independent solutions of (1.3) and  $W = u\dot{v} - v\dot{u}$  is their Wronskian (constant). There is a vast literature devoted to this equation and many applications (for example, see [5, 6] and [7] for other references).

The  $\mathfrak{sl}(2, \mathbb{R})$  symmetry algebra of (1.2) has the basis

$$X_1 = u^2\partial_t + u\dot{u}x\partial_x, \quad X_2 = uv\partial_t + \frac{1}{2}(\dot{u}v + u\dot{v})x\partial_x, \quad X_3 = v^2\partial_t + v\dot{v}x\partial_x. \quad (1.5)$$

They satisfy the commutation relations

$$[X_1, X_2] = WX_1, \quad [X_1, X_3] = 2WX_2, \quad [X_2, X_3] = WX_3. \quad (1.6)$$

$W$  can be normalized to unity by scaling the elements of the algebra. On introducing the point transformation

$$\tau = \int \frac{dt}{u^2}, \quad \xi = \frac{x}{u}, \quad v = W\tau u \quad (1.7)$$

we obtain, up to scaling, the canonical form of (1.5)

$$\tilde{X}_1 = \partial_\tau, \quad \tilde{X}_2 = \tau\partial_\tau + \frac{1}{2}\xi\partial_\xi, \quad \tilde{X}_3 = \tau^2\partial_\tau + \tau\xi\partial_\xi \quad (1.8)$$

satisfying the same commutation relations in (1.6) with  $W = 1$ . The canonical algebra (1.8) is one of four inequivalent representatives of Lie's classification of the algebra  $\mathfrak{sl}(2, \mathbb{R})$  on the real plane. The  $SL(2, \mathbb{R})$  group action on the plane  $(\tau, \xi)$  is

$$(\tau, \xi) \rightarrow \left( \frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1}\xi \right), \quad ad - bc = 1. \quad (1.9)$$

Action on  $\tau$  induces a Möbius transformation. Transformation (1.7) takes Eq. (1.2) to  $\xi''(\tau) = q\xi(\tau)^{-3}$  which is invariant under (1.8).

We shall briefly review the particular case  $p = q = 1$ ,  $u(t) = \cos t$ ,  $v(t) = \sin t$  ( $W = 1$ ) which frequently arises in applications. In this case, (1.2) is invariant under the  $\mathfrak{sl}(2, \mathbb{R})$  symmetry algebra

$$\begin{aligned} X_1 &= \cos^2 t \partial_t - \sin t \cos t x \partial_x, & X_2 &= \sin t \cos t \partial_t + \frac{1}{2}(\cos^2 t - \sin^2 t) x \partial_x, \\ X_3 &= \sin^2 t \partial_t + \sin t \cos t x \partial_x. \end{aligned} \quad (1.10)$$

From (1.7) we find that under the transformation  $\tau = \tan t$ ,  $\xi = x \sec t$ , the algebra (1.10) is transformed to the equivalent representation (1.8). Moreover, by the change of basis  $X_1 + X_3 \rightarrow X_1$ ,  $X_1 - X_3 \rightarrow X_3$ ,  $2X_2 \rightarrow X_2$  we have another representation of (1.10)

$$X_1 = \partial_t, \quad X_2 = \sin 2t \partial_t + \cos 2t x \partial_x, \quad X_3 = \cos 2t \partial_t - \sin 2t x \partial_x.$$

From (1.7) and (1.9) it follows that the  $\text{SL}(2, \mathbb{R})$  action on solutions is given by

$$\tilde{x}(\tilde{t}) = [(au(\tilde{t}) - cv(\tilde{t}))^2 + (bu(\tilde{t}) - dv(\tilde{t}))^2]^{1/2} x(t), \quad \tilde{\tau} = \tan \tilde{t} = \frac{a\tau + b}{c\tau + d}, \quad (1.11)$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}),$$

and states that  $\tilde{x}(\tilde{t})$  is a solution whenever  $x(t)$  is a solution. This transformation formula can be expressed as

$$\tilde{x}(\tilde{t}) = [Au^2 + 2Buv + Cv^2]^{1/2} x(t), \quad (1.12)$$

where we have redefined

$$A = a^2 + b^2, \quad B = -(ac + bd), \quad C = c^2 + d^2$$

so that  $AC - B^2 = (ad - bc)^2 = 1$ . By (1.12), the constant solution  $x = x_0 = q^{1/4}$  is transformed to the general solution

$$x(t) = x_0 [Au^2 + 2Buv + Cv^2]^{1/2}, \quad (1.13)$$

where  $AC - B^2 = 1$ . This solution can be reconciled with (1.4) by redefining the group parameters as  $(A, B, C) \rightarrow q^{1/2}(A, B, C)$  so that  $AC - B^2 \rightarrow q$  and the factor  $x_0$  has been set to unity. The reader is referred to Section 2 for an alternative derivation of the general solution in the general case (1.2).

In view of the previous discussion, our first objective is to identify a class of equations (1.1) invariant under a one-dimensional subgroup of the symmetry group of the EP equation. Under this condition, the equation can be reduced to an equivalent autonomous equation by means of a canonical transformation. This study is done in Section 2. In Section 3, the most general set of nonlinear differential equations invariant under the symmetry found in Section 2 is identified. Section 4 explores a generalization of the associated Ermakov-Lewis invariant. A related power transformation is considered in Section 5, rediscovering in this way the nonlinear superposition rule of the Kummer-Schwarz equation. Finally, Section 6 is devoted to show some practical applications of the exposed theory to concrete models of interest in Classical and Quantum Mechanics and Geometry.

## 2 Symmetries of a class of equations with singular term

In this section we analyze the Lie point symmetries of Eq. (1.1). A vector field of the form

$$X = \tau(t, x)\partial_t + \xi(t, x)\partial_x \quad (2.1)$$

will generate a Lie point symmetry of the equation if the second prolongation  $\text{pr}^{(2)}X$  of the vector field  $X$  is annihilated on solutions of (1.1). Forming this infinitesimal invariance condition and splitting the resulting equation we obtain a system of linear differential equations for the coefficients  $\tau$  and  $\xi$ . Solving this system we find that  $\tau(t, x) = a(t)$ ,  $\xi(t, x) = c(t)x$  such that

$$q(2c - \dot{a}) = 0, \quad 2\dot{c} = \ddot{a} \quad (2.2)$$

and moreover  $a(t)$  and  $c(t)$  are linked to the coefficients  $p(t), g(t)$  of the equation by the remaining determining equations

$$(m-1)cg + 2g\dot{a} + a\dot{g} = 0, \quad \ddot{c} + 2p\dot{a} + a\dot{p} = 0. \quad (2.3)$$

From (2.2), if  $q = 0$  then  $c(t) = \frac{1}{2}\dot{a}(t) + C_0$ , where  $C_0$  is an integration constant. In [1, 2] the case  $C_0 \neq 0$  ( $q = 0$ ,  $m = 3$ ) was discussed, leading to an autonomous equation with friction, which is not integrable in general. In our case we assume  $q \neq 0$ , or  $q = C_0 = 0$ , so that our vector field turns out to be

$$X(a) = a(t)\partial_t + \frac{1}{2}\dot{a}(t)x\partial_x, \quad a \neq 0, \quad (2.4)$$

where  $a$  must satisfy the conditions

$$M(a) = \ddot{a} + 4p\dot{a} + 2\dot{p}a = 0, \quad (2.5)$$

and

$$2a\dot{g} + (m+3)\dot{a}g = 0. \quad (2.6)$$

From Eq. (2.6)  $g(t)$  is completely determined by  $a(t)$  in the form

$$g(t) = g_0 a^{-(m+3)/2}, \quad (2.7)$$

where  $g_0$  is an integration constant. We assume  $m \neq -3$  for otherwise our equation is reduced to the standard Ermakov-Pinney equation. On the other hand, the third order equation (2.5) is known to have a maximal symmetry algebra of dimension 7 for third order linear equations with an  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra [8, 9]. The structure of the algebra is  $\mathfrak{g} = (\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}) \triangleright \mathfrak{a}(3)$ , where  $\mathfrak{a}(3)$  is a three dimensional abelian ideal. The  $\mathfrak{sl}(2, \mathbb{R})$  subalgebra emerges from the fact that it has a solution basis  $\{u^2, uv, v^2\}$  with  $u$  and  $v$  being two linearly independent solutions of the time-dependent oscillator equation (1.3). In other words, the general solution of (2.5) is

$$a = Au^2 + 2Buv + Cv^2 \quad (2.8)$$

for arbitrary coefficients  $A, B, C$ . On the other hand, Eq. (2.5) has the first integral ( $a$  is an integrating factor)

$$K = \frac{1}{4}(2a\ddot{a} - \dot{a}^2) + pa^2. \quad (2.9)$$

If we substitute  $a$  from (2.8) into (2.9) we find that the constants  $A, B, C$  should be constrained by  $K = (AC - B^2)W^2$ . Moreover, the substitution  $a = y^2(t)$  transforms (2.9) to the EP equation

$$\ddot{y} + py = Ky^{-3}. \quad (2.10)$$

So we have reconfirmed that the general solution of (2.10) is given by the formula  $y = \sqrt{a(t)}$ .

For  $g$  having the form (2.7) the original equation can be written as

$$\ddot{x} + px = qx^{-3} + g_0x^m a^{-(m+3)/2} \quad (2.11)$$

admitting a one-dimensional subalgebra of the symmetry algebra of the usual Ermakov-Pinney (EP) equation. Using the coordinate transformation (reparametrization of  $t$  and linear change of  $x$ )

$$\tau = \int \frac{dt}{a(t)}, \quad y = \frac{x}{\sqrt{a}} \quad (2.12)$$

we can write the autonomous form of (2.11) as

$$\frac{d^2y}{d\tau^2} + Ky = qy^{-3} + g_0y^m. \quad (2.13)$$

Although Eq. (2.13) is exactly integrable (in the sense that it is reducible to quadratures) its full integration (general solution) requires non-elementary functions (See Section 4 for its first integral).

However, it is possible to construct group-invariant solutions as particular solutions. From the requirement of the invariant curve condition we must have the special solutions, up to a multiplicative nonzero constant which can be absorbed into  $a(t)$ ,  $x(t) = \sqrt{a(t)}$ , where  $a > 0$  is given by (2.8). This solution actually corresponds to the constant solution  $y = 1$  of (2.13) So, under the constraint

$$K = (AC - B^2)W^2 = q + g_0, \quad (2.14)$$

the invariant solution of Eq. (2.11) will have the form

$$x(t) = [Au^2 + 2Buv + Cv^2]^{1/2}. \quad (2.15)$$

In view of (2.14), at most two parameters among  $A, B, C$  can be specified so that all the parameters figuring in solution (2.15) have been fixed and consequently Eq. (2.15) will represent only one particular solution.

We propose a simple example to illustrate the previous arguments. For  $m = 1$  Eq. (2.11) is reduced to the EP equation with a modified potential

$$\ddot{x} + (p - g_0a^{-2})x = qx^{-3}.$$

The choice  $p = 0$  gives the equation

$$\ddot{x} - g_0(A + 2Bt + Ct^2)^{-2}x = qx^{-3}, \quad K = AC - B^2 = q + g_0$$

with the special solution  $x = (A + 2Bt + Ct^2)^{1/2}$ .

The structure of the symmetry algebra for a more general variant of the EP equation (2.10) suggests that it can be of interest to construct general second order nonlinear ordinary differential equations invariant under at least one-dimensional subgroup of the symmetry group of the EP equation.

### 3 The most general class of invariant equations

One can construct most general nonlinear ordinary differential equation (ODE) of any order invariant under the symmetry (2.4) for which particular group-invariant solutions can be found or reduction of order can be effected. In order to construct second order invariant ODE we need to find a set of functionally independent second order differential invariants of the vector field (2.4), where  $a(t)$  lies in the span of the solution set  $\{u^2, uv, v^2\}$ . Invariants are found by solving the first order PDE defined by the condition  $\text{pr}^{(2)}X(I) = 0$ ,  $I = I(t, x, \dot{x}, \ddot{x})$ , where

$$\text{pr}^{(2)}X(a) = X(a) + \frac{1}{2}(\ddot{a}x - \dot{a}\dot{x})\partial_{\dot{x}} + \frac{1}{2}(\ddot{\ddot{a}}x - 3\dot{a}\ddot{x})\partial_{\ddot{x}}. \quad (3.1)$$

We find the first order invariants as

$$I_1 = \frac{x}{\sqrt{a}}, \quad I_2 = \sqrt{a} \left( \dot{x} - \frac{\dot{a}}{2a}x \right) = a\dot{I}_1. \quad (3.2)$$

A second order differential invariant can be obtained by solving the ODE

$$\frac{dt}{a} = \frac{d\ddot{x}}{(\ddot{\ddot{a}}x - 3\dot{a}\ddot{x})/2} \quad (3.3)$$

as

$$a^{3/2}[\ddot{x} - \frac{1}{4a^2}(2a\ddot{a} - \dot{a}^2)x] = \text{const}. \quad (3.4)$$

Using the first integral (2.9) we find the second order invariant  $I_3 = a^{3/2}(\ddot{x} + px)$ . An alternative to the construction of  $I_3$  is to form the differentiation  $dI_2/dI_1$ . In general, two functionally invariant is sufficient to determine other higher order invariants by the process of invariant differentiation  $(D_x^k I_2)/(D_x^k I_1)$ . The most general equation admitting an invariant solution of the form  $x = \sqrt{a}$ ,  $a = Au^2 + 2Buv + Cv^2$  becomes  $H(I_1, I_2, I_3) = 0$  with  $H$  an arbitrary function. A special form is

$$\ddot{x} + px = a^{-3/2}F(I_1, I_2), \quad (3.5)$$

where  $F$  is an arbitrary function of the first two invariants. For the special choice  $F = qI_1^{-3} + g_0I_1^m$  we recover Eq. (1.1) with  $F(1) = q + g_0$ . Another choice  $F = -\alpha I_2 + f(I_1)$ ,  $\alpha$  a constant, produces the invariant equation

$$\ddot{x} + \alpha a^{-1}\dot{x} + (p - \frac{\alpha}{2}a^{-2}\dot{a})x = a^{-3/2}f(a^{-1/2}x)$$

admitting the same solution formula (2.15):  $x = \sqrt{a}$  provided  $(AC - B^2)W^2 = f(1)$ . In general, the same rule applies to Eq. (3.5) with the constraint  $M(a) = K = (AC - B^2)W^2 = F(1, 0)$ . Note that no choice of the invariant  $I_1$  can introduce dependence on time in the coefficient of the nonlinearity  $x^{-3}$ .

One can of course include higher order invariants to construct higher order non-linear ODEs. For example, a third order invariant is given by

$$I_4 = a^{5/2}[\ddot{x} + \frac{d}{dt}(px)] + \frac{3}{2}\dot{a}I_3$$

and the corresponding third order equation would have the form  $H(I_1, I_2, I_3, I_4) = 0$ . A subclass that does not depend explicitly on  $a$  and  $\dot{a}$  is picked out by the invariant equation  $J_2 = F(J_1)$ , where  $F$  is an arbitrary function and  $J_1, J_2$  are the invariants defined in terms of  $I_i$  ( $i = 1, 2, 3, 4$ ) by

$$J_1 = I_1^3 I_3 = x^3(\ddot{x} + px), \quad J_2 = I_1^4(I_1 I_4 + 3I_2 I_3) = x^4[x(\ddot{x} + \dot{p}x) + (3\dot{x}\ddot{x} + 4px\dot{x})]. \quad (3.6)$$

The subclass  $J_2 = F(J_1)$  and the EP equation with basis (1.5) enjoy the same  $\mathfrak{sl}(2, \mathbb{R})$  symmetry structure.  $x = \sqrt{a}$  will be a solution to this class only when  $F(K) = 0$ ,  $K = (AC - B^2)W^2$ . Contrary to the second order case, this solution provides only a two-parameter particular solution since the equation is of third order. On the other hand, a two dimensional solvable subalgebra of the nonsolvable  $\mathfrak{sl}(2, \mathbb{R})$  algebra can be used to reduce its integration to a pair of quadratures and the solution of a Riccati equation.

The change of variable  $y = x^2$  simplifies  $J_1, J_2$  into a more familiar form

$$\tilde{J}_1 = \frac{1}{4}(2y\ddot{y} - \dot{y}^2) + py^2, \quad \tilde{J}_2 = \frac{y^2}{2}M(y) = \frac{y^2}{2}(\ddot{y} + 4p\dot{y} + 2\dot{p}y).$$

For example, in the special case  $p = 0$  ( $a \in \text{span}\{1, t, t^2\}$ ) we get the invariant equation  $J_2 = F(J_1)$ , where  $J_1 = 2y\ddot{y} - \dot{y}^2$ ,  $J_2 = y^2\ddot{y}$ . In terms of the invariants  $r = \dot{y}$  and  $s = J_1 = 2y\ddot{y} - \dot{y}^2$  of the solvable subalgebra  $\{\partial_t, t\partial_t + y\partial_y\}$  it is reduced to the Riccati equation

$$\frac{dr}{ds} = \frac{s + r^2}{4F(s)}.$$

Once this equation is solved for  $r = R(s)$  and inverted as  $s = S(r)$  we can integrate it by two quadratures using the fact that  $s = S(r)$  is invariant under the above two-dimensional subalgebra.

We note that by the change of variable  $y = x^{-2}$  the invariants  $J_1 = x^3\ddot{x}$ ,  $J_2 = x^4[x\ddot{x} + 3\dot{x}\dot{x}]$  take the form

$$\tilde{J}_1 = -y^{-6}[y^2\ddot{y} - 6y\dot{y}\ddot{y} + 6\dot{y}^2], \quad \tilde{J}_2 = -\frac{1}{2}y^{-2}\left[\frac{\ddot{y}}{y} - \frac{3}{2}\left(\frac{\dot{y}}{y}\right)^2\right].$$

The corresponding third order invariant equation is

$$y^2\ddot{y} - 6y\dot{y}\ddot{y} + 6\dot{y}^2 = y^6F(s), \quad s = y^{-2}\left[\frac{\ddot{y}}{y} - \frac{3}{2}\left(\frac{\dot{y}}{y}\right)^2\right]. \quad (3.7)$$

Eq. (3.7) appeared in [10] as the canonical third order equation invariant under one of the three inequivalent planar actions of the group  $SL(2, \mathbb{C})$ . The authors of [10] showed that the general solution can be formulated in parametric form in terms of solutions of the linear Schrödinger equation  $\psi''(\omega) - 1/2F(\omega)\psi(\omega) = 0$ , in addition to the alternative derivation based on Lie reduction method. An interesting fact that is worth mentioning is that the symmetry algebra of Eq. (3.7) spanned by

$$\{\partial_t, t\partial_t - y\partial_y, t^2\partial_t - 2ty\partial_y\} \quad (3.8)$$

is connected to the first prolongation of the Lie algebra  $\{\partial_t, t\partial_t, t^2\partial_t\}$  of the first action of  $SL(2, \mathbb{C})$  on the plane  $(t, x)$ , which projects to the plane  $(t, y)$ ,  $y = \dot{x}$ . We emphasize that all  $\mathfrak{sl}(2, \mathbb{R})$  actions appearing throughout this paper are actually locally isomorphic to (3.8).

## 4 Ermakov-Lewis invariants

Using the invariants  $y = I_1$ ,  $w = I_2$  we can reduce the order of (3.5) by one. Taking into account (2.9) we can write (3.5) in the form

$$w \frac{dw}{dy} + Ky = F(y, w). \quad (4.1)$$

In general integration of (4.1) depends on the form of  $F(y, w)$  which can be intractable. For example,  $F(y, w) = \beta(y)w + \gamma(y)$  leads to an Abel equation of the second kind. However, integrable cases (for example when  $F = \alpha(y)w^2$  we integrate a Bernoulli's equation) can generate a family of first integrals (invariants). In particular, if  $F$  is independent of  $w$ , the above equation is separable and can be integrated easily to give the first integral

$$I = \frac{1}{2}[w^2 + Ky^2] - \hat{F}(y) = \text{const.}, \quad \hat{F}'(y) = F(y) \quad (4.2)$$

or

$$I = \frac{1}{2}[(\sqrt{a}\dot{x} - \frac{1}{2}\frac{\dot{a}}{\sqrt{a}}x)^2 + (AC - B^2)\frac{x^2}{a} - \hat{F}(\frac{x}{\sqrt{a}})], \quad (4.3)$$

where  $a$  is in the linear span of the set  $\{u^2, uv, v^2\}$ . This can be viewed as a generalization of the Ermakov-Lewis invariant [11, 12]. For example, choosing  $a(t) = u^2$  ( $A = 1, B = C = 0, K = AC - B^2 = 0$ ) we find

$$I = \frac{1}{2}(u\dot{x} - \dot{u}x)^2 - \hat{F}\left(\frac{x}{u}\right) = \text{const.} \quad (4.4)$$

This invariant actually coincides with a special case of the one for the so-called Ermakov system [13] defined by

$$\ddot{x} + p(t)x = x^{-3}F\left(\frac{x}{u}\right), \quad \ddot{u} + p(t)u = u^{-3}G\left(\frac{x}{u}\right) \quad (4.5)$$



when  $G = 0$ . This special system corresponds exactly to (3.5) obtained by the replacement  $F \rightarrow I_1^{-3} F(I_1)$  for  $a = u^2$ . For the special choice  $F(y) = qy^{-3}$  we recover the Ermakov-Lewis invariant

$$I = \frac{1}{2}(u\dot{x} - \dot{u}x)^2 + \frac{q}{2}\left(\frac{u}{x}\right)^2,$$

which was originally derived by eliminating  $p(t)$  between the equations

$$\ddot{x} + px = qx^{-3}, \quad \ddot{u} + pu = 0,$$

and using the integrating factor  $u\dot{x} - \dot{u}x$ .

For the special case  $F = qI_1^{-3} + g_0I_1^m$  (Eq. (1.1)) the Ermakov-Lewis invariant is

$$I = \frac{1}{2}w^2 + \frac{K}{2}y^2 + \frac{q}{y^2} - \frac{g_0}{m+1}y^{m+1}, \quad m \neq -1$$

and

$$I = \frac{1}{2}w^2 + \frac{K}{2}y^2 + \frac{q}{y^2} - g_0 \log y$$

for  $m = -1$ .

Let us mention that the canonical coordinates (2.12) straightening out the vector field (2.4) to  $\tilde{X} = \partial_\tau$  converts the invariant equation (3.5) into the autonomous form

$$\frac{d^2y}{d\tau^2} + (AC - B^2)W^2y = F\left(y, \frac{dy}{d\tau}\right).$$

In terms of  $\partial_\tau$  invariants  $y$  and  $w = dy/d\tau$  we recover Eq. (4.1) (reduction of order by one).

## 5 Equivalent equations and second order Kummer-Schwarz equation

We wish to relate equations invariant under the vector fields (2.4) to other equations with the same symmetry by some power transformations of  $x$ . To this end we consider vector fields

$$X(a) = a(t)\partial_t + k\dot{a}(t)z\partial_z, \quad k \neq 0, \quad (5.1)$$

where  $a$  is assumed as in Section 3 and  $k$  is a real constant. Vector field (2.4) is equivalent to (5.1) under the transformation  $z = x^{2k}$ . Second order differential invariants of (5.1) are found as

$$J_1 = a^{-k}z, \quad J_2 = a^{-k}(a\dot{z} - k\dot{a}z), \quad J_3 = a^{2-k}(\ddot{z} + 2kpz) - (2k-1)[\dot{a}J_2 + \frac{k}{2}\dot{a}^2J_1]. \quad (5.2)$$

Note that in view of (2.5) we have

$$\text{pr}^{(2)}X(J_3) = ka^{2-k}xM(a) = 0.$$

The corresponding invariant equation will have the form  $H(J_1, J_2, J_3) = 0$ . Solving for  $J_3$  gives the class

$$a^{2-k}(\ddot{z} + 2kpz) - (2k-1)[\dot{a}J_2 + \frac{k}{2}\dot{a}^2J_1] = F(J_1, J_2), \quad (5.3)$$

where  $F$  is an arbitrary function. Thus, specifying two of the free parameters involved in  $a$ , say,  $A, C$  (then  $B$  is fixed by the following condition) within the family of invariant equations (5.3) we find the following formula for their particular (invariant) solutions

$$z(t) = a(t)^k = (Au^2 + 2Buv + Cv^2)^k, \quad 2k(AC - B^2)W^2 = F(1, 0). \quad (5.4)$$

Formula (5.4) can give rise to a general superposition rule only when Eq. (5.3) is independent of the function  $a$  and its derivative  $\dot{a}$  such that the three-parameter function  $a$  appears only in the symmetry transformation. It is easy to see that this is the case for a subclass of (5.3) when  $k = 1/2$  and  $F = qJ_1^{-3}$ . This choice leads to the Ermakov-Pinney equation (1.2). It is interesting to observe that there is another choice of the parameter  $k$  and the function  $F$  which makes the functions  $a$  and  $\dot{a}$  disappear in (5.3). This indeed happens when  $k = -1$  and

$$F(J_1, J_2) = \frac{3}{2}\frac{J_2^2}{J_1} - 2qJ_1^3, \quad q \neq 0,$$

which leads to the remarkable second order Kummer-Schwarz (2KS) equation

$$\ddot{z} = \frac{3}{2}\frac{\dot{z}^2}{z} + 2pz - 2qz^3. \quad (5.5)$$

Eq. (5.5) arises as particular instance of the second order Gambier equation. See [14] for the details. Apparently its  $\mathfrak{sl}(2, \mathbb{R})$  symmetry algebra (locally isomorphic to (3.8)) will be generated by the vector fields

$$X_1 = u^2\partial_t - 2u\dot{u}z\partial_z, \quad X_2 = uv\partial_t - (u\dot{v} + v\dot{u})z\partial_z, \quad X_3 = v^2\partial_t - 2v\dot{v}z\partial_z. \quad (5.6)$$

They satisfy the commutation relations (1.6). We can relate the EP equation (1.2) to the 2KS equation (5.5) by the power transformation  $x = z^{-1/2}$ ,  $z > 0$ . Apparently, the invariant solution turns into the general solution. Hence, using our approach we have rediscovered that Eq. (5.5) admits as its general solution the superposition rule

$$z = (Au^2 + 2Buv + Cv^2)^{-1}, \quad (AC - B^2)W^2 = q.$$

This fact was established in [14, 15] in the framework of Lie systems and quasi-Lie schemes.

2KS equation is related to a class of second order ODEs whose general solution has been obtained as a special case by the method of homogenous functions in [16]. The transformation  $z = w^{(n-1)/2}$ , where  $n \neq 1$  is a real parameter, maps Eq. (5.5) equation to

$$\ddot{w} + \frac{4p}{1-n}w = \frac{n+3}{4}\frac{\dot{w}^2}{w} + \frac{4q}{1-n}w^n, \quad q \neq 0 \quad (5.7)$$

and preserves the  $\mathfrak{sl}(2, \mathbb{R})$  symmetry of the 2KS equation with generators

$$X_1 = u^2 \partial_t + 2ku\dot{u}w\partial_w, \quad X_2 = uv\partial_t + k(u\dot{v} + v\dot{u})w\partial_w, \quad X_3 = v^2 \partial_t + 2kv\dot{v}w\partial_w, \quad (5.8)$$

where  $k = 2/(1 - n)$ . We observe that Eq. (5.7) can be recovered by forming the invariant equation

$$J_3 = F(J_1, J_2) = \sigma \frac{J_2^2}{J_1} + \frac{4q}{1 - n} J_1^n, \quad q \neq 0, \quad n \neq 1$$

for suitable choices of  $k$  and  $\sigma$ . This equation can be written in the form

$$\ddot{z} + 2kpz = \sigma \frac{\dot{z}^2}{z} + Q(t)\dot{z} + R(t)z + \frac{4q}{1 - n} a^{k-2-nk} z^n \quad (5.9)$$

for some functions  $Q(t)$  and  $R(t)$  whose exact forms will be determined below. The coefficient of  $z^n$  dictates that we must choose  $k = 2/(1 - n)$ . For this value of  $k$  we have

$$Q(t) = \frac{\dot{a}}{a} \frac{(n + 3 - 4\sigma)}{1 - n}, \quad R(t) = -\frac{\dot{a}^2}{a^2} \frac{(n + 3 - 4\sigma)}{(1 - n)^2}, \quad n \neq 1.$$

We require  $\sigma = (n + 3)/4$  for the absence of  $a$  and  $\dot{a}$  in the coefficients  $Q$  and  $R$ . Hence, Eq. (5.9) boils down to Eq. (5.7).

From (5.4) we obtain the general solution of (5.7)

$$w = (Au^2 + 2Buv + Cv^2)^{2/(1-n)}, \quad (AC - B^2)W^2 = q. \quad (5.10)$$

The special case  $n = -3$  of (5.7) is the EP equation. The transformation that takes (5.7) to the EP equation

$$\ddot{x} + px = qx^{-3}$$

is given by  $w = x^{4/(1-n)}$ .

In passing we recall that the following ODE admits the 8-dimensional  $\mathfrak{sl}(3, \mathbb{R})$  algebra, in other words it is linearizable by point transformations if and only if  $n = \sigma$  or  $n = 1$  (equivalent to  $g(t) = 0$ ) (This can be seen by Lie-Tresse linearization test.):

$$\ddot{w} + pw = \sigma \frac{\dot{w}^2}{w} + g(t)w^n. \quad (5.11)$$

Finally, we note that for the choice  $k = 1$  ( $n = -1$ ) in (5.7) we find the  $\mathfrak{sl}(2, \mathbb{R})$  invariant equation

$$\ddot{w} = \frac{1}{2} \frac{\dot{w}^2}{w} - 2pw + \frac{2q}{w}, \quad (5.12)$$

which is nothing more than Eq. (2.9) with  $a$  and  $K$  replaced by  $w$  and  $q$ , respectively. From (5.10) the general solution of (5.12) becomes

$$w = Au^2 + 2Buv + Cv^2, \quad (AC - B^2)W^2 = q.$$

The transformation  $w = x^2$  establishes again the connection of (5.12) with the above EP equation.

## 6 Applications

The purpose of this section is to emphasize the interest of the Lie point symmetry analysis performed in the previous sections by showing some examples of applications to models coming from different scientific fields. More concretely, we will see that Eq. (1.1) covers a wide range of interesting situations.

### 6.1 Central force fields

The motion of particle in a central force field is ruled by the system

$$\ddot{u} = f(t, |u|) \frac{u}{|u|}, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (6.1)$$

In a central force field every orbit is planar, hence we can assume  $N = 2$  without loss of generality. Then, passing to polar coordinates  $u(t) = x(t)e^{i\theta(t)}$ ,  $x > 0$  system (6.1) is equivalent to

$$\ddot{x} = c^2 x^{-3} + f(t, x), \quad (6.2)$$

where  $c = x^2 \dot{\theta}$  is the *angular momentum* of  $u$ , which is a constant of motion. For  $f(t, x) = g(t)x^m - p(t)x$ , we recover Eq. (1.1) with  $q = c^2$ . Once the scalar equation (6.2) is solved, the angular variable is found by a simple integration in  $\dot{\theta} = cx^{-2}$ . Let us write explicitly the equation

$$\ddot{x} + p(t)x = c^2 x^{-3} + g(t)x^m. \quad (6.3)$$

We have identified a whole family of integrable equations of this type. The most general way to proceed is to fix an arbitrary function  $a(t)$  and define  $g(t)$  by (2.7) and

$$p(t) = \frac{c^2 + g_0}{a^2} - \frac{1}{2} \left[ \frac{\ddot{a}}{a} - \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 \right].$$

This formula for  $p(t)$  is obtained from (2.9) and the restriction (2.14). For this choice of the coefficients, Eq. (6.3) is integrable and has  $\sqrt{a(t)}$  as a particular solution and the coordinate transformation (2.12) transforms the equation into the autonomous form (2.13) with  $K = c^2 + g_0$ , which has the energy as a conserved quantity and hence the orbits are simply the level curves of the energy.

### 6.2 Modulated amplitude waves in Bose-Einstein condensates

The 1D Gross-Pitaevskii equation

$$iu_t = -\frac{1}{2}u_{xx} + V(x)u + h(x)|u|^2u \quad (6.4)$$

models the evolution of a quasi-one-dimensional Bose-Einstein condensate (BEC) subjected to a external magnetic trapping  $V(x)$  and a particle interaction  $g(x)$ , which

is tunable by Feshbach resonance. Assuming that both functions are  $L$ -periodic, a modulated amplitude wave (see [17] and the references therein) is a doubly periodic solution (in space and time), described by the ansatz

$$u(x, t) = R(x) \exp(i[\theta(x) - \mu t]). \quad (6.5)$$

Entering the ansatz (6.5) into (6.4) and taking real and imaginary parts, the amplitude  $R(x)$  follows the second order equation

$$\ddot{R}(x) = \frac{c^2}{R^3} + 2(V(x) - \mu)R + 2h(x)R^3, \quad (6.6)$$

where the parameter  $c$  is a conserved quantity given by the relation

$$R^2(x)\dot{\theta}(x) = c, \quad (6.7)$$

in total analogy with the conservation of angular momentum of a particle under a central force field, shown in the last subsection. In fact, renaming the variables  $R \rightarrow x, x \rightarrow t$ , we arrive at

$$\ddot{x} + 2(\mu - V(t))x = \frac{c^2}{x^3} + 2h(t)x^3, \quad (6.8)$$

which is exactly the same equation (1.1) with  $m = 3$ ,  $q = c^2$ ,  $p(t) = 2(\mu - V(t))$  and  $g(t) = 2h(t)$ . Now, we can proceed as before. Of course, fixing a periodic  $a(t)$  we obtain periodic coefficients. Other way to proceed is to fix  $V(t) \equiv 0$ , then  $u = \cos \omega t, v = \sin \omega t$  with  $\omega = \sqrt{2\mu}$  is a fundamental system of  $\ddot{x} + 2\mu x = 0$ . Taking  $A = 2, B = 0, C = 1$  in (2.8), we get  $a(t) = 1 + \cos^2 \omega t$  and  $K = 2\omega^2 = 4\mu$ . Then, by (2.7) and (2.14), we can fix

$$h(t) = (\omega^2 - \frac{c^2}{2})a^{-3}.$$

For such coefficients, (6.8) has the periodic solution  $x(t) = [1 + \cos^2 \omega t]^{1/2}$  and it is reducible by the change (2.12) to the autonomous form

$$\ddot{y} + 2\omega^2 y = \frac{c^2}{y^3} + (2\omega^2 - c^2)y^3.$$

### 6.3 The method of moments for a multi-dimensional BEC

For the  $n$ -dimensional Gross-Pitaevskii equation with parabolic trap and time-dependent coefficients

$$iu_t = -\frac{1}{2}\Delta u + \frac{1}{2}\lambda(t)^2|x|^2u + g(t)|u|^{2p}u, \quad (6.9)$$

the method of moments (see [18] and the references therein) analyses the evolution of certain integral quantities with physical meaning. In particular, the second momentum (or variance)

$$I(t) = \int_{\mathbb{R}} |x|^2 |u|^2 dx$$

represents the width of the wave packet. It is proved in [18] that  $x(t) = \sqrt{I}$  verifies the second order equation

$$\ddot{x} + \lambda(t)^2 x = q_1 x^{-3} + q_2 g(t) x^m,$$

where  $q_1, q_2$  are certain conserved quantities and  $m = -(np + 2)/2$ .

## 6.4 The 2-dimensional $L_p$ Minkowski problem

Geometrically, the  $L_p$  Minkowski problem [19] asks about conditions on a function  $g(x) \in C(\mathbb{S}^{n-1})$  that guarantee that it is the  $L_p$  surface area measure of a convex body. When  $n = 2$ , the problem is reduced to find a  $2\pi$ -periodic solution of the equation

$$x'' + x = g(t)x^{p-1}.$$

The same equation arises in the anisotropic curve shortening problem [20]. Observe that in this case  $q = 0$ . Now we can proceed as in Subsection 6.2 to construct explicit cases of solvability. For example, for  $p = -4$  the equation

$$\ddot{x} + x = \frac{2(1 + \cos^2 t)}{x^5}$$

is integrable.

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